

Covariant path integrals and black holes

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Abstract

The thermal nature of the propagator in a collapsed black-hole spacetime is shown to follow from the non-trivial topology of the configuration space in tortoise coordinates by using the path integral formalism.

The path integral (PI) formalism is useful to calculate propagators in configuration spaces (CS) endowed with a non-trivial topology, such as in curved spacetimes. Moreover, even if the CS topology is trivial, this may not be the case for its Euclidian section, where the PI should always be computed. For example, in an eternal black-hole background or in Rindler spacetime, although the CS itself has a trivial topology in Kruskal or Rindler coordinates, the Euclidian CS has a periodic structure in the Euclidian time-like coordinate. In these spacetimes endowed with an event-horizon (EH), it can be shown that the thermal properties of the propagator follows from the periodic structure of the Euclidian CS [1–4].

In a collapsing black-hole spacetime, however, this periodic structure is missing. One has to find in this case another procedure to obtain the thermal properties of the propagator. It is the purpose of this paper to show that a similar periodic structure may be recovered in tortoise coordinates if one requires that they cover the entire spacetime by allowing them to take complex values. In the resulting complex CS, the EH has a cylinder-like topology. The propagator is then obtained from a PI by adding all the contributions of the classes of

path with different winding number around the EH [5]. An advantage of this procedure is that the contributions of the paths crossing the EH are also included in the PI, which is not the case if the Euclidianisation is performed in Kruskal coordinates.

One of the simplest models for gravitational collapse is the Synge model, where a Schwarzschild black hole is created from an imploding spherical shell of radiation [6]. We shall concentrate our attention on the spacetime region \mathcal{R} where the thermal radiation emitted by the black hole may be detected by an inertial observer. This region is defined to be located outside the imploding shell, far from the black hole and at late times. If x denotes the Kruskal coordinates, this region is defined by $x^+ \gg 1$ and $x^- \approx x_H^-$, where $x_H^- \equiv x_0^+ - 4M$ is the position of the EH. In this region, the line element is given by

$$ds^2 \approx \frac{dx^+ dx^-}{\kappa (x_H^- - x^-)} - r(x^+, x^-)^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (1)$$

where $\kappa = (4M)^{-1}$, and the tortoise coordinates y are defined by

$$x^+(y^+) \approx y^+, \quad x^-(y^-) \approx x_H^- - 2e^{-\kappa(y^- - x_H^-)}, \quad (2)$$

where $(y^0, y^1) = (t, r + 2M \ln |r - 2M|)$, if t and r are the Schwarzschild coordinates. The Kruskal CS is given by

$$\mathcal{K} = \{ (x^+, x^-, \theta, \phi) \in \mathbb{R}^2 \times \mathbb{S}^2 \mid r(x^+, x^-) \geq 0 \}, \quad (3)$$

and its homotopic properties are trivial.

The tortoise coordinates cover only the region $x^- < x_H^-$ outside the EH, and so cannot be used to parameterise a path crossing the EH. Instead of introducing another set of coordinates to cover this region, one extends tortoise coordinates to *complex* values, to keep a sense of connectedness for the CS. This is possible since $x^-(y^- + i\pi/\kappa) > x_H^-$, and thus the complex number $y^- + i\pi/\kappa$ describes a point inside the EH. The CS \mathcal{T} in tortoise coordinates is thus defined as the complex preimage of the CS \mathcal{K} through the transformation $x = x(y)$. Since the transformation $x^- = x^-(y^-)$ is periodic in the imaginary direction, a point $x \in \mathcal{K}$ has a infinite number of complex preimages denoted by y^ν , where $\nu \in \mathbb{Z}$. They are given by $(y^\nu)^+ = x^+$ and

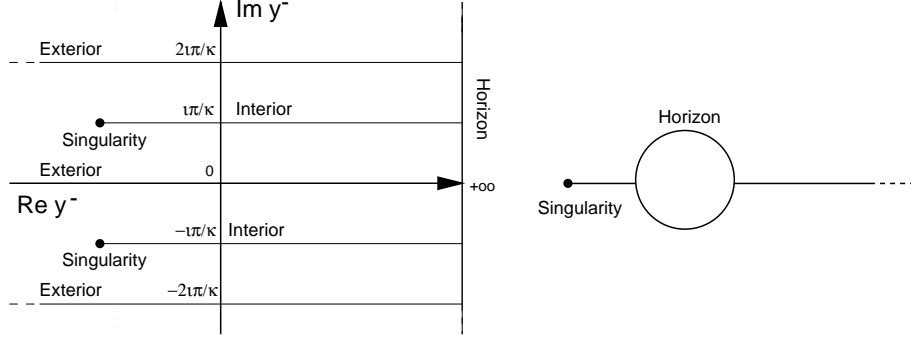


FIG. 1. A section of the tortoise complex covering space (on the left) and of the configuration space \mathcal{T} (on the right) along the y^- coordinate.

$$(y^\nu)^- = \begin{cases} y^- + i2\pi(\nu + 1/2)/\kappa, & \text{if } x^- > x_H^-, \\ y^- + i2\pi\nu/\kappa, & \text{if } x^- < x_H^-, \end{cases} \quad (4)$$

where y^- is the real preimage of $x_H^- - |x_H^- - x^-|$. The complex CS \mathcal{T} is thus given by

$$\mathcal{T} = \{ (y^+, y^-, \theta, \phi) \in \mathbb{R} \times \left[\bigcup_{\mu \in \mathbb{Z}} R_{\mu/2} \bigcup T \right] \times \mathbb{S}^2 \mid r(y^+, y^-) \geq 0 \}, \quad (5)$$

where $R_\mu = \mathbb{R} + i2\pi\mu/\kappa$ and $T = +\infty + i\mathbb{R}$ (the set T parameterises the EH). The CS \mathcal{T} is multiply connected and is represented in fig. 1. The topology of the EH in \mathcal{T} is given by $\mathbb{R} \times \mathbb{S}_{1/\kappa}^1 \times \mathbb{S}^2$, where $\mathbb{S}_{1/\kappa}^1$ is a circle of radius κ^{-1} . The fundamental group of \mathcal{K} is thus isomorphic to \mathbb{Z} . The classes of paths are labelled by the integer ν which is the winding number of the paths around the circle $\mathbb{S}_{1/\kappa}^1$.

In quantum mechanics, the amplitude to move from an initial point x_i to a final point x_f in a parameter-time $s = s_f - s_i$ is given by the propagator $K(x_i, x_f; s)$, or heat kernel. It is defined as a PI with endpoints x_i and x_f , and is written symbolically as

$$K_{\text{Kruskal}}(x_i, x_f; s) = \sum_{g, x(\cdot) \in \mathcal{K}} e^{\frac{i}{\hbar} S_g[x(\cdot)]}, \quad (6)$$

where \sum_g and S_g are the covariant sum over paths and action respectively [7]. The set of paths on which the sum is taken is fixed by the CS \mathcal{K} and defines the *Kruskal vacuum*. If both endpoints x_i and x_f belong to the region \mathcal{R} , one may try replacing the metric g

appearing in the path integral *everywhere* by the metric of Eq. (1). The weight of each path that ventures out of the region \mathcal{R} is changed, but we may conjecture that this will not greatly affect the propagator in \mathcal{R} , since this is a local quantity. The simplicity of choice of the metric (1), which is the Minkowski metric η in tortoise coordinates, allows us to compute the propagator easily. Because the propagator is a biscalar, one writes

$$\sum_{x(\cdot) \in \mathcal{K}} e^{\frac{i}{\hbar} S_g[x(\cdot)]} \approx \sum_{y(\cdot) \in \mathcal{T}} e^{\frac{i}{\hbar} S_\eta[y(\cdot)]}, \quad (7)$$

and thus the sum may be taken over paths in the complex tortoise CS. The PI may be rewritten by summing over the classes of paths. The contribution of the entire class of paths with winding number ν gives the free propagator \widehat{K}_0 (in tortoise coordinates) with arguments y_i and y_f' . The total propagator between y_i and y_f (both in region \mathcal{R}) is given by

$$\widehat{K}_{\text{Kruskal}}(y_i; y_f; s) \approx \sum_{\nu \in \mathbb{Z}} \widehat{K}_0(y_i^+, y_i^-, \Omega_i; y_f^+, y_f^- + i2\pi\nu/\kappa, \Omega_f; s), \quad (8)$$

and represents an outgoing thermal flux of particles with temperature $T = \hbar\kappa/(2\pi k)$.

We have shown that the homotopic properties of the configuration space are not an intrinsic feature of the black-hole spacetime, but depend on the set of coordinates chosen to cover it. Since the inverse of the transformation relating Kruskal and tortoise coordinates is not analytic, the CS \mathcal{K} and \mathcal{T} have different topologies. These properties have no physical consequences on the classical motion of a particle. However, in a quantum mechanical framework, a particle may tunnel across the horizon, and the topology of the whole configuration space is then physically relevant. The radiation of a black hole is thermal because, from the point of view of a distant inertial observer, there is a denumerably infinite number of ways for a particle to tunnel through the horizon.

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